Question 1 (10 Marks) START A NEW PAGE **Marks**

(a) Find the gradient of the tangent to the curve $y = e^{\sin x}$ at $x = \dfrac{\pi}{3}$. **2**

(b) Differentiate with respect to x.

(i) $\cot 3x$. **2**

(ii) $\ln\left(\dfrac{x}{x^2+1}\right)$. **2**

(c) Express $0.4\dot{6}$ as an geometric series and hence write its equivalent fraction in simplest terms. **2**

(d) Show that $y = e^{\frac{1}{x}}$ has no stationary points. Justify your answer. **2**

Question 2 (10 Marks) START A NEW PAGE

(a) Evaluate $\displaystyle\int_1^2 \frac{x^2-9}{x^3}\,dx$ **2**

(b) Find

(i) $\displaystyle\int \tan^2(2x-1)\,dx$. **2**

(ii) $\displaystyle\int \frac{3x}{x-1}\,dx$. **2**

(c) A weather balloon released at ground level 2100m from an observer rises at a rate of 100 metres per minute.

(i) Show that, after t minutes, the height of the balloon, h metres, is given by $h = 2100\tan\theta$, where θ is the angle of elevation. **1**

(ii) Find the rate, in radians per minute, at which the angle of elevation of the observer is increasing when the balloon is at an altitude of 1400 metres. **3**

Question 3 (10 Marks) **START A NEW PAGE** **Marks**

(a) The nth term of a series is given by $T_n = n + \left(\frac{1}{3}\right)^n$. **4**

Find the sum of the first 10 terms.

(b) Prove, by mathematical induction for $n = 1, 2, 3, \ldots$ and $x > 0$, **4**

$$\frac{1}{x(x+1)^n} = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2} - \ldots - \frac{1}{(x+1)^n}.$$

(c) Given $f(x) = 2^{3x}$, find $f'(0)$. **2**

Question 4 (10 Marks) **START A NEW PAGE**

(a) (i) The inner surface of a bowl is formed by rotating the curve $y = \ln(x - 2)$ about the y – axis between $y = 0$ and $y = 2$. Calculate the volume of water that this bowl holds when the depth of water is filled to a depth of h units. **3**

(ii) If water is poured into the bowl at a rate of 50 cubic units per second, find the *exact* rate at which the water level is rising when the depth of water is 1.5 units. **2**

(b) (i) Express $\sqrt{3}\cos 2x - \sin 2x$ in the form of $A\cos(2x + \alpha)$ for $0 < \alpha < \frac{\pi}{2}$ and $A > 0$. **2**

(ii) Hence, or otherwise, sketch the curve $y = \sqrt{3}\cos 2x - \sin 2$; $0 \le x \le 2\pi$. **2**

(iii) Find the first positive solution for $\sqrt{3}\cos 2x - \sin 2x = 1$. **1**

Question 5 (10 Marks) **START A NEW PAGE** **Marks**

(a) (a) Given that $\sum_{k=1}^{n} \frac{k}{(k+1)!} = 1 - \frac{1}{(n+1)!}$, show that **3**

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!} + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+2)!}$$

(b) Sketch the graph of $y = ln[2x(3 - x)]$, showing all important features. **4**

(c) Find $\int \frac{x^2}{\sqrt{9 - x^2}} dx$ using the substitution $\alpha = \sin^{-1}\left(\frac{x}{3}\right)$. **3**

Question 6 (10 Marks) **START A NEW PAGE**

(a) Calculate the sum of the series **4**

$$2^{5x} + 2^{3x} + 2^{x} + \ldots + 2^{(11 - 6k)x}.$$

(b) Amy borrowed \$280 000 to purchase an apartment at a reducible interest rate of 7.8 % per annum compounded monthly from 2006 from her mortgage lender. She makes equal monthly repayments of \$*M* until the loan is repaid.

(i) How much did Amy owe after her first monthly repayment of \$*M*? **1**

(ii) Calculate Amy's equal monthly instalment \$*M*, if her loan is for 25 years. **2**

(iii) After 1 year, the Reserve Bank increases the interest rate by 0.25% *pa*, however, her mortgage lender increases her interest rate by a further 0.1% *pa*. **3**
How much extra does Amy need to pay, per month, with the total extra interest rate rise?

THE END

Question 1 (10 Marks) **START A NEW PAGE** **Marks**

(a) $\frac{dy}{dx} = \cos x e^{\sin x}$ *when* $x = \frac{\pi}{3}$, $\frac{dy}{dx} = \frac{1}{2} e^{\frac{\sqrt{3}}{2}}$ 2

(b) (i) $\frac{d}{dx}\cot(3x) = -1(\tan(3x)^{-2} \times 3\sec^2(3x)$ 2

$= -\frac{3\sec^2 3x}{\tan^2 3x} = -3\operatorname{cos}ec^2 3x$

(ii) $\frac{d}{dx}\ln\left(\frac{x}{x^2+1}\right) = \frac{d}{dx}\left(\ln x - \ln\left(x^2+1\right)\right) = \frac{1}{x} - \frac{2x}{x^2+1}$ 2

$= \frac{-x^2+1}{x(x^2+1)}$

(c) $0.4\dot{6} = 0.4 + 0.06666... = 0.4 + 0.06 + 0.006 + 0.0006 + ...$ 2

$A = 0.06,\ r = 0.1$ ➔ $S_\infty = \frac{0.06}{1-0.1} = \frac{1}{15}$

$\therefore\ 0.4\dot{6} = \frac{4}{10} + \frac{1}{15} = \frac{7}{15}$

(d) $y' = -\frac{e^{\frac{1}{x}}}{x^2}$. Now $e^{\frac{1}{x}} \neq 0$ as $x \neq 0$. 2

Question 2 (10 Marks) **START A NEW PAGE**

(a) $\int_1^2 \frac{1}{x} - 9x^{-3}dx = \left[\ln x + \frac{9}{2x^2}\right]_1^2 = \ln 2 - \frac{27}{8}$ 2

(b) $\int \tan^2(2x-1)dx$. 2

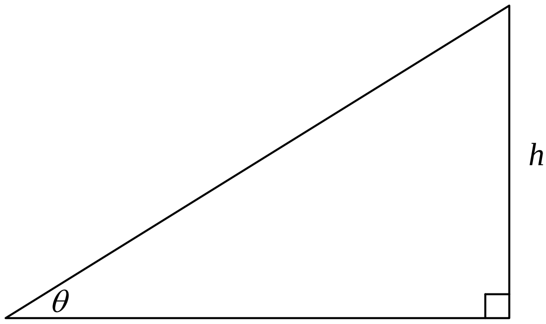
$= \int [\sec^2(2x-1) - 1]dx$

$= \frac{1}{2}\tan(2x-1) - x + c$

(ii) $\int \frac{3x}{x-1}dx$ 2

$= \int \frac{3(x-1)+3}{x-1}dx = \int 3 + \frac{3}{x-1}dx = 3x + 3\ln(x-1) + c$

(c)



$\frac{dh}{dt} = 100$ 4 mks

And $\frac{h}{2100} = \tan\theta$

$\therefore\ h = 2100\tan\theta$

$\frac{dh}{d\theta} = 2100\sec^2\theta$

Now $\frac{d\theta}{dt} = \frac{d\theta}{dh}\cdot\frac{dh}{dt}$

$\therefore \frac{d\theta}{dt} = \frac{\cos^2\theta}{2100}\times 100$

But when h = 1400, $\tan\theta = \frac{2}{3}$ ➔ $\cos^2\theta = \frac{9}{13}$

$\therefore \frac{d\theta}{dt} = \frac{9}{21\times 13} = \frac{3}{91}$ radian per minute.

<u>Question 3 (10 Marks)</u> **START A NEW PAGE**

(a) $T_1 = 1 + 1/3$, $T_2 = 2 + 1/9$, $T_3 = 3 + 1/27$, …. 4

$\therefore$ we have an AP + GP sum.

$\therefore$ AP sum = 1 +2+3+…+10 = 55

GP sum = $1/3 + 1/9 + 1/27 + \ldots + 1/3^{10}$

$$= \frac{\frac{1}{3}\left(1-\left(\frac{1}{3}\right)^{10}\right)}{\frac{2}{3}} = \frac{1}{2}\left(1-\frac{1}{3^{10}}\right)$$

$\therefore$ total sum = $55 + \frac{1}{2}\left(1-\frac{1}{3^{10}}\right)$

(b) **Test *n* = 1,** 4

LHS = $\frac{1}{x(x+1)^1}$ RHS = $\frac{1}{x} - \frac{1}{x+1}$

$= \frac{x+1-x}{x(x+1)^1}$

$= \frac{1}{x(x+1)}$

$\therefore$ true for $n = 1$, $x > 0$

Assume true for *n* = *k*

$$\frac{1}{x(x+1)^k} = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2} - \ldots - \frac{1}{(x+1)^k}$$

For *n*=*k* + 1

$$\frac{1}{x}-\frac{1}{x+1}-\frac{1}{(x+1)^2}-\ldots-\frac{1}{(x+1)^k}-\frac{1}{(x+1)^{k+1}}$$

$$=\frac{1}{x(x+1)^k}-\frac{1}{(x+1)^{k+1}}$$ *using the assumption*

$$=\frac{x+1-x}{x(x+1)^{k+1}}$$

$$=\frac{1}{x(x+1)^{k+1}}$$

$\therefore$ Since true for $n = 1$ and proven true for $n = k+1$, assuming true for n = k
It is also true for all $n \geq 1$.

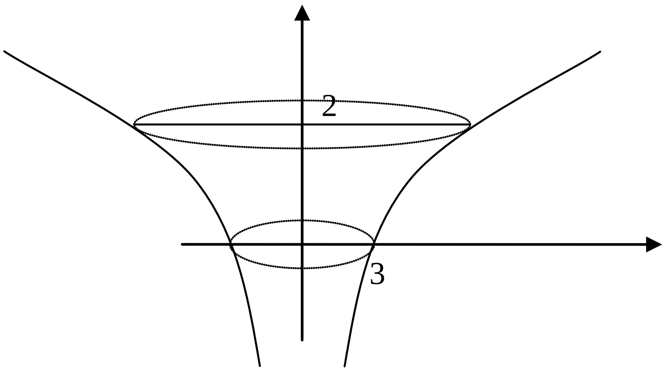
(c) $f(x) = 2^{3x} = 8^x$ 2

$f'(x) = \ln 8.8^x$

$\therefore \ f'(0) = \ln 8$

Question 4 (10 Marks) **STARTA NEW PAGE** **Marks**

(a) (i) 3



Vol = $\pi\int_0^h \left(2+e^y\right)^2 dy \quad = \pi\left[4+4e^y+\frac{1}{2}e^{2y}\right]_0^h \quad = \pi\left[4h+4e^h+\frac{1}{2}e^{2h}-4-\frac{1}{2}\right]$

$= \pi\left[4h+4e^h+\frac{1}{2}e^{2h}-\frac{9}{2}\right] \ u^3.$

(ii) $\frac{dV}{dt}=50, \ \frac{dV}{dh}=\pi\left(4+4e^h+e^{2h}\right)$ 2

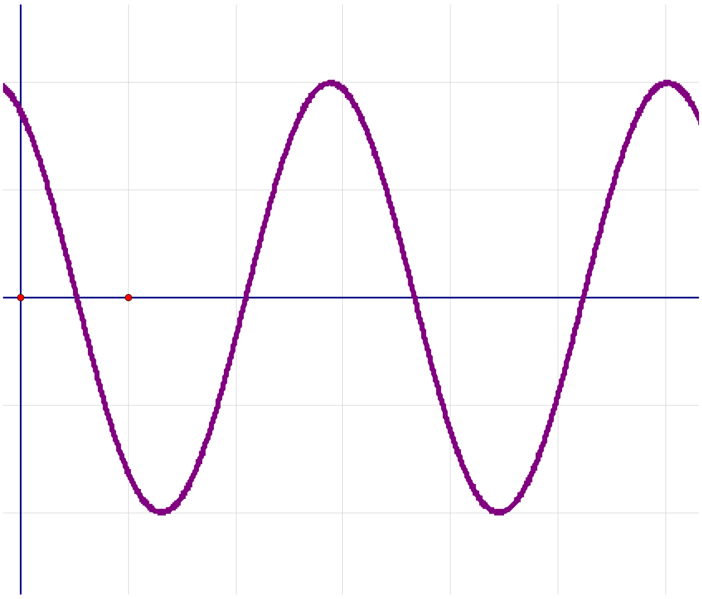
$\therefore \ 50 = \pi\left(4+4e^{1.5}+e^3\right)\cdot\frac{dh}{dt}$ $\quad\therefore \ \frac{dh}{dt}=\frac{50}{\pi\left(4+4e^{1.5}+e^3\right)}$ units/sec.

(b) (i) $\sqrt{3}\cos 2x - \sin 2x \equiv A[\cos 2x \cos\alpha - \sin 2x \sin\alpha]$

$\therefore \ A = \sqrt{3+1} = 2$, $\cos\alpha = \sqrt{3}$, $\sin \alpha = 1$ $\quad \therefore \ \tan \alpha = \frac{1}{\sqrt{3}} \ \therefore \ \alpha = \frac{\pi}{6}$ 2

$\therefore \quad \sqrt{3}\cos 2x - \sin 2x = 2\cos(2x + \frac{\pi}{6})$

(ii) Let $y = 2\cos(2x + \frac{\pi}{6})$ 2



(iii) $2\cos(2x + \frac{\pi}{6}) = 1$ 1

$\cos(2x + \frac{\pi}{6}) = \frac{1}{2}$

$\therefore\ 2x + \frac{\pi}{6} = \frac{\pi}{3} \qquad \therefore\ 2x = \frac{\pi}{6} \qquad \therefore\ x = \frac{\pi}{12}$

<u>Question 5 (10 Marks)</u> **START A NEW PAGE**

(a) $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!} + \frac{n+1}{(n+2)!} = \sum_{k=1}^{n} \frac{k}{(k+1)!} + \frac{n+1}{(n+2)!}$ 3

$= 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!}$

$= 1 - \frac{n+2}{(n+2)!} + \frac{n+1}{(n+2)!}$

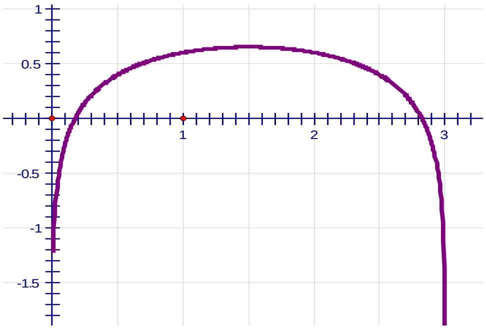
$= 1 + \frac{-n-2+n+1}{(n+2)!} = 1 - \frac{1}{(n+2)!}$

(b) $y = \ln 2x + \ln(3 - x)$; $x \neq 0$, 3 4

$y' = \frac{1}{x} + \frac{-1}{3-x} = \frac{3-2x}{x(3-x)} \qquad \therefore$ tp at $x = 1.5$, $y = \ln(4.5)$

Test tp

$y'' = \frac{-1}{x^2} + \frac{-1}{(3-x)^2} = -\frac{4}{9} - \frac{4}{9} = -\frac{8}{9} < 0 \quad \therefore$ absolute max since only 1 tp.



NOTE: Curve is asymptotic at $x = 0$ and $x = 3$.

(c) $x = 3\sin\alpha \therefore dx = 3\cos\alpha \, d\alpha$ 3

$$\therefore \int \frac{9\sin^2\alpha \, 3\cos\alpha}{\sqrt{9 - 9\sin^2\alpha}} d\alpha = \int \frac{27\sin^2\alpha\cos\alpha}{3\cos\alpha} d\alpha = \int 9\left(\frac{1-\cos 2\alpha}{2}\right) d\alpha$$

$$= \frac{9}{2}\left(\alpha - \frac{1}{2}\sin 2\alpha\right) + c$$

<u>Question 6 (10 Marks)</u> **START A NEW PAGE**

(a) Let $S = 2^{5x} + 2^{3x} + 2^{x} + \ldots + 2^{(11\text{-}6k)x}$ and as a GS $r = 2^{-2x}$ 4

So $2^{-2x}.S = 2^{3x} + 2^{5x} + 2^{7x} + \cdots + 2^{(11-6k)x} + 2^{(9-6k)x}$.

$S(1 - 2^{-2x}) = 2^{5x} - 2^{(9-6k)x}$.

$$\therefore \quad S = \frac{2^{5x} - 2^{(9-6k)x}}{1 - 2^{-2x}} = \frac{2^{7x} - 2^{(11-6k)x}}{2^{2x} - 1}.$$

Note the number of terms was $N = 3k$-2.

(b) (i) $R = \frac{13}{2000}$ per month. (0.0065) 1

280 000×1.0065 – *M*

(ii) Month 1: 280 000×1.0065 – M 2

Month 2: $280\,000 \times 1.0065^2 - M(1 + 1.0065)$

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Month 300: $280000 \times 1.0065^{300} - M(1 + 1.0065 + 1.0065^2 + \ldots + 1.0065^{299})$

$$\therefore M = \frac{280000 \times 1.0065^{300}}{1(1.0065^{300} - 1) \div 0.0065}$$

= **$2 124.12**

(iii)Amount remaining after 1 year = $280\,000 \times 1.0065^{12} - 2124.12 \times \frac{(1 \cdot 0065^{12} - 1)}{0 \cdot 0065}$. 3

= $276 217.22

New interest rate = 8.15%

$$M_1 = \frac{276\,217.22 \times \left(\frac{24163}{24000}\right)^{288}}{\left(\left(\frac{24163}{24000}\right)^{288} - 1\right) \div \frac{163}{24000}} = \$2187.37$$

$\therefore$extra payments with the extra 0.35% is $2187.37 - $2 124.12 = **$63.25**